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Estimation Methods of Three Parameters Discrete Generalized Exponential Distribution

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Abstract

This article presents the estimation methods of three parameters Discrete Generalized Exponential distribution. The two-parameter generalized exponential distribution was introduced by Gupta and Kundu (1999). We present the three parameters discrete Generalized Exponential distribution is the sum of infinite probability function. Moment estimation, inverse integer moment estimation, moment generating function, maximum likelihood estimation and L-moment estimation are derived for this infinite probability function.

Keywords: Three parameter Generalized Exponential distribution, Sum of infinite probability function, Moment estimation, moment generating function, maximum likelihood estimation and L-moment estimation.

1. Introduction

The Generalized Exponential models are the reliability models can be used in the reliability engineering discipline. This paper focuses on the reliability analysis of the discrete Generalized Exponential distribution's to model in which some operational time has already been accumulated for the equipment of interest. This paper presents the relationship between shape parameter and other properties such as probability function, cumulative distribution function, reliability function, hazard function, cumulative hazard function, r_{th} moment estimation, Inverse integer moment estimation, moment generating functions, maximum likelihood estimation and L-moment estimation are presented mathematically. The Discrete Generalized Exponential distribution will be suitable for modeling for the applications of mechanical or electrical components lying in the life testing experiment. Some works has already been done on Generalized Exponential distribution by Gupta and Kundu (2003). Debasis Kundu, Rameshwar D. Gupta and Anubhav Manglick (2005) presented the Discriminating between the log-normal and generalized exponential distributions. Gupta, R. D; Kundu, D (2001 b) also derived Generalized exponential distributions for different methods of estimation.

2. Discrete Generalized Exponential Models

The Discrete Generalized Exponential probability distribution has three parameters η, β and x_0 . It can be used to represent the failure probability density function (PDF) is given by:

$$f_{GEP}(x)=\frac{\beta}{\eta}\sum_{m=0}^{\beta-1}C_{m:\beta-1}(-1)^{m}Exp\left[-\left(\frac{(x-x_0)(1+m)}{\eta}\right)\right],\eta>0,\beta>0,x_0>0,-\infty<x_0<x \qquad (2.1)$$

Where $C_{m:\beta-1}=\frac{(\beta-1)!}{m!(\beta-m-1)!}$, β is the shape parameter representing the different pattern of the discrete Generalized Exponential PDF and is positive and η is a scale parameter representing the characteristic life, t_0 is a location parameter and sometimes called a guarantee time, failure-free time or minimum life. If x_0 =0 then the three parameter discrete Generalized Exponential distribution is said to be two-parameter discrete Generalized Exponential distribution.

The cumulative distribution function (CDF) of Discrete Generalized Exponential distribution is denoted by $F_{GEP}(x)$ and is defined as

$$F_{GEP}(x)=\frac{\beta}{1+m}\sum_{m=0}^{\beta-1}C_{m:\beta-1}(-1)^{m}\left[1-Exp\left(-\left(\frac{(x-x_0)(1+m)}{\eta}\right)\right)\right] \qquad (2.2)$$

When the CDF of the Generalized Exponential distribution has zero value then it represents no failure components by x_0 .

The reliability function (RF), denoted by $R_{GEP}(x)$ is also known as the survivor function and is defined as l- $F_{GEP}(x)$

$$R_{GEP}(x)=1-\frac{\beta}{1+m}\sum_{m=0}^{\beta-1}C_{m:\beta-1}(-1)^{m}\left[1-Exp\left(-\left(\frac{(x-x_0)(1+m)}{\eta}\right)\right)\right] \qquad (2.3)$$

The hazard function (HF) is also known as instantaneous failure rate denoted by $h_{GEP}(x)$ and is defined as $f_{GEP}(x)/\ R_{GEP}(x)$

$$h_{GEP}(x)=\frac{\frac{\beta}{\eta}\sum_{m=0}^{\beta-1}C_{m:\beta-1}(-1)^{m}Exp\left[-\left(\frac{(x-x_0)(1+m)}{\eta}\right)\right]}{1-\frac{\beta}{1+m}\sum_{m=0}^{\beta-1}C_{m:\beta-1}(-1)^{m}\left[1-Exp\left(-\left(\frac{(x-x_0)(1+m)}{\eta}\right)\right)\right]} \qquad (2.4)$$

The cumulative hazard function (CHF), denoted by $H_{GEP}(x)$ and is defined as:

$$H_{GEP}(x)=\ln\left|\frac{\beta}{1+m}\sum_{m=0}^{\beta-1}C_{m:\beta-1}(-1)^{m}\left[1-Exp\left(-\left(\frac{(x-x_0)(1+m)}{\eta}\right)\right)\right]\right| \qquad (2.5)$$

3. rth Moment Estimation

The rth moment of Discrete Generalized Exponential distribution about origin is given by

$$\mu_r' = \int_{x_0}^{\infty} x^r \frac{\beta}{\eta} \sum_{m=0}^{\beta-1} C_{m:\beta-1} (-1)^m Exp\left[-\left(\frac{(x-x_0)(1+m)}{\eta} \right) \right] dx \tag{3.1}$$

Put x_0 =0, Then the above reliability model will provide the rth moment of Generalized Exponential Distribution

$$\mu_r' = \beta\eta^r \sum_{m=0}^{\beta-1} C_{m:\beta-1} \frac{(-1)^m \Gamma(r+1)}{(1+m)^{r+1}}, \quad r = 1,2,3,4 \tag{3.2}$$

The special cases of these rth moments of Generalized Exponential Distribution are

$$\mu_1' = \beta\eta \sum_{m=0}^{\beta-1} C_{m:\beta-1} \frac{(-1)^m}{(1+m)^2} \tag{3.2a}$$

$$\mu_2' = \beta\eta^2 \sum_{m=0}^{\beta-1} C_{m:\beta-1} \frac{2(-1)^m}{(1+m)^3} \tag{3.2b}$$

$$\mu_3' = \beta\eta^3 \sum_{m=0}^{\beta-1} C_{m:\beta-1} \frac{6(-1)^m}{(1+m)^{r+1}} \tag{3.2c}$$

$$\mu_4' = \beta\eta^4 \sum_{m=0}^{\beta-1} C_{m:\beta-1} \frac{24(-1)^m}{(1+m)^5} \tag{3.2d}$$

The variance, skewness and kurtosis measures can now be calculated for the rth moments about mean of Discrete Generalized Exponential distribution using the relations

$$Var(x) = \beta\eta^2 \sum_{m=0}^{\beta-1} C_{m:\beta-1} \frac{2(-1)^m}{(1+m)^3} - \left(\beta\eta \sum_{m=0}^{\beta-1} C_{m:\beta-1} \frac{(-1)^m}{(1+m)^2} \right)^2$$

$$Skewness(x) = \frac{\beta\eta^3 \sum_{m=0}^{\beta-1} C_{m:\beta-1} \frac{6(-1)^m}{(1+m)^{r+1}} - 3\left(\beta\eta \sum_{m=0}^{\beta-1} C_{m:\beta-1} \frac{(-1)^m}{(1+m)^2} \right)\left(\beta\eta^2 \sum_{m=0}^{\beta-1} C_{m:\beta-1} \frac{2(-1)^m}{(1+m)^3} \right) + 2\left(\beta\eta \sum_{m=0}^{\beta-1} C_{m:\beta-1} \frac{(-1)^m}{(1+m)^2} \right)^3}{\left(\beta\eta^2 \sum_{m=0}^{\beta-1} C_{m:\beta-1} \frac{2(-1)^m}{(1+m)^3} - \left(\beta\eta \sum_{m=0}^{\beta-1} C_{m:\beta-1} \frac{(-1)^m}{(1+m)^2} \right)^2 \right)^{\frac{3}{2}}}$$

$$Kurtosis(x)=\frac{\beta\eta^4\sum_{m=0}^{\beta-1}C_{m:\beta-1}\frac{24(-1)^m}{(1+m)^5}-4\left(\beta\eta\sum_{m=0}^{\beta-1}C_{m:\beta-1}\frac{(-1)^m}{(1+m)^2}\right)\left(\beta\eta^3\sum_{m=0}^{\beta-1}C_{m:\beta-1}\frac{6(-1)^m}{(1+m)^{r+1}}\right)+6\left(\beta\eta^2\sum_{m=0}^{\beta-1}C_{m:\beta-1}\frac{2(-1)^m}{(1+m)^3}\right)\left(\beta\eta\sum_{m=0}^{\beta-1}C_{m:\beta-1}\frac{(-1)^m}{(1+m)^2}\right)^2-3\left(\beta\eta\sum_{m=0}^{\beta-1}C_{m:\beta-1}\frac{(-1)^m}{(1+m)^2}\right)^4}{\left(\beta\eta^2\sum_{m=0}^{\beta-1}C_{m:\beta-1}\frac{2(-1)^m}{(1+m)^3}-\left(\beta\eta\sum_{m=0}^{\beta-1}C_{m:\beta-1}\frac{(-1)^m}{(1+m)^2}\right)^2\right)^2}$$

4. Inverse rth Moment Estimation

The Inverse rth moment estimation of Discrete Generalized Exponential distribution about origin is given by

$$\mu'_{r^{-1}}=\int_{x_0}^{\infty}x^{-r}\frac{\beta}{\eta}\sum_{m=0}^{\beta-1}C_{m:\beta-1}(-1)^m Exp\left[-\left(\frac{(x-x_0)(1+m)}{\eta}\right)\right]dx \tag{4.1}$$

Put x_0 =0, Then the above reliability model will provide the inverse rth moment of Discrete Generalized Exponential distribution is

$$\mu'_{r^{-1}}=\beta\eta^{-r}\sum_{m=0}^{\beta-1}C_{m:\beta-1}\frac{(-1)^m\Gamma(1-r)}{(1+m)^{1-r}},\ r=1,2,3,4 \tag{4.2}$$

The special cases of these inverse rth moments of Discrete Generalized Exponential distribution are

$$\mu'_{1^{-1}}=\beta\eta^{-1}\sum_{m=0}^{\beta-1}C_{m:\beta-1}(-1)^m \tag{4.2a}$$

$$\mu'_{2^{-1}}=\beta\eta^{-2}\sum_{m=0}^{\beta-1}C_{m:\beta-1}(-1)^m(1+m)\Gamma-1 \tag{4.2b}$$

$$\mu'_{3^{-1}}=\beta\eta^{-3}\sum_{m=0}^{\beta-1}C_{m:\beta-1}(-1)^m(1+m)^2\Gamma-2 \tag{4.2c}$$

$$\mu'_{4^{-1}}=\beta\eta^{-4}\sum_{m=0}^{\beta-1}C_{m:\beta-1}(-1)^m(1+m)^3\Gamma-3 \tag{4.2d}$$

The variance, skewness and kurtosis measures can now be calculated for the Inverse rth moments about mean of Discrete Generalized Exponential distribution using the relations

$$Var(x^{-1}) = \beta\eta^{-2}\sum_{m=0}^{\beta-1} C_{m:\beta-1}(-1)^m (1+m)\Gamma - 1 - \left(\beta\eta^{-1}\sum_{m=0}^{\beta-1} C_{m:\beta-1}(-1)^m\right)^2$$

$$Skewness(x^{-1}) = \frac{\beta\eta^{-3}\sum_{m=0}^{\beta-1} C_{m:\beta-1}(-1)^m(1+m)^2\Gamma - 2 - 3\left(\beta\eta^{-1}\sum_{m=0}^{\beta-1} C_{m:\beta-1}(-1)^m\right)\left(\beta\eta^{-2}\sum_{m=0}^{\beta-1} C_{m:\beta-1}(-1)^m(1+m)\Gamma - 1\right) + 2\left(\beta\eta^{-1}\sum_{m=0}^{\beta-1} C_{m:\beta-1}(-1)^m\right)^3}{\left(\beta\eta^{-2}\sum_{m=0}^{\beta-1} C_{m:\beta-1}(-1)^m(1+m)\Gamma - 1 - \left(\beta\eta^{-1}\sum_{m=0}^{\beta-1} C_{m:\beta-1}(-1)^m\right)^2\right)^{\frac{3}{2}}}$$

$$Kurtosis(x^{-1}) = \frac{\begin{array}{l}\beta\eta^{-4}\sum_{m=0}^{\beta-1} C_{m:\beta-1}(-1)^m(1+m)^3\Gamma - 3 - 4\left(\beta\eta^{-1}\sum_{m=0}^{\beta-1} C_{m:\beta-1}(-1)^m\right)\left(\beta\eta^{-3}\sum_{m=0}^{\beta-1} C_{m:\beta-1}(-1)^m(1+m)^2\Gamma - 2\right) + \\ 6\left(\beta\eta^{-2}\sum_{m=0}^{\beta-1} C_{m:\beta-1}(-1)^m(1+m)\Gamma - 1\right)\left(\beta\eta^{-1}\sum_{m=0}^{\beta-1} C_{m:\beta-1}(-1)^m\right)^2 - 3\left(\beta\eta^{-1}\sum_{m=0}^{\beta-1} C_{m:\beta-1}(-1)^m\right)^4\end{array}}{\left(\beta\eta^{-2}\sum_{m=0}^{\beta-1} C_{m:\beta-1}(-1)^m(1+m)\Gamma - 1 - \left(\beta\eta^{-1}\sum_{m=0}^{\beta-1} C_{m:\beta-1}(-1)^m\right)^2\right)^2}$$

5. Moment generating function

$$M_x(t) = \int_{x_0}^{\infty} e^{tx}\frac{\beta}{\eta}\sum_{m=0}^{\beta-1} C_{m:\beta-1}(-1)^m Exp\left[-\left(\frac{(x-x_0)(1+m)}{\eta}\right)\right]dx \tag{5.1}$$

Put x_0 =0, Then the above reliability model will provide the moment generating function of Discrete Generalized Exponential distribution is

$$M_x(t) = \sum_{m=0}^{\beta-1} C_{m:\beta-1}\frac{(-1)^m\beta}{(1+m)-t\eta} \tag{5.2}$$

Then the rth moments about origin of Discrete Generalized Exponential distribution is defined as

$$\mu_r' = \frac{d^r}{dt^r}\sum_{m=0}^{\beta-1} C_{m:\beta-1}\frac{(-1)^m\beta}{(1+m)-t\eta} \tag{5.3}$$

For finding these moments about origin the above function will take the values $r = 1,2,3,4$

At the time $t_0 = 0$, will provide $\mu_{1,}'\mu_2', \mu_3'$ and μ_4'. Using these moments about origin, then we find variance, skewness and kurtosis measures can now be calculated for the rth moments about mean of Discrete Generalized Exponential Distribution.

6. Maximum Likelihood Estimation

We consider estimation by the method of maximum likelihood for the Generalized Exponential Distribution. The log likelihood for a random sample $x_1, x_2, ..., x_n$ from (2.1) taking $x_0 = 0$ is

$$\log L(\beta,\eta) = n\log\beta - n\log\eta + (\beta-1)\sum_{i=1}^{n}\log\left(1-\exp(-\frac{x}{\eta})\right) - \frac{1}{\eta}\sum_{i=1}^{n}x_i \tag{6.1}$$

The first order derivatives of (6.1) with respect to the two parameters are

$$\frac{\partial \log L}{\partial\beta} = \frac{n}{\beta} + \sum_{i=1}^{n}\log\left(1-\exp(-\frac{x}{\eta})\right) \tag{6.2}$$

$$\frac{\partial \log L}{\partial\eta} = \frac{-n}{\eta} - \frac{(\beta-1)}{\eta^2}\sum_{i=1}^{n}\frac{x\exp(-\frac{x}{\eta})}{1-\exp(-\frac{x}{\eta})} + \frac{1}{\eta^2}\sum_{i=1}^{n}x_i \tag{6.3}$$

By setting these expressions equal to zero in (6.2) and (6.3), then solving them simultaneously yield the maximum likelihood estimates of the two parameters.

$$\hat{\beta} = \frac{n}{\sum_{i=1}^{n}\log\left(1-\exp(-\frac{x}{\hat{\eta}})\right)^{-1}} \tag{6.4}$$

$$\hat{\eta} = \frac{1}{n}\left[(1-\beta)\sum_{i=1}^{n}\frac{x\exp(-\frac{x}{\eta})}{1-\exp(-\frac{x}{\eta})} + \sum_{i=1}^{n}x_i\right] \tag{6.5}$$

Minimum Variance Bound (MVB) for the discrete Generalized Exponential distribution is

$$-E\left(\frac{\partial^2 \log L}{\partial\beta^2}\right) = \frac{n}{\beta^2} \tag{6.6}$$

$$-E\left(\frac{\partial^2 \log L}{\partial\beta\partial\eta}\right) = \frac{n\beta}{\eta}\sum_{m=0}^{\beta-2}\frac{C_{\beta-2:m}(-1)^m}{(m+2)^2} \tag{6.7}$$

$$-E\left(\frac{\partial^2 \log L}{\partial \beta^2}\right)=\frac{n}{\eta^2}\left(2\beta\sum_{m=0}^{\beta-1}\frac{C_{\beta-1:m}(-1)^m}{(m+1)^2}-1\right)-$$

$$\frac{\beta(1-\beta)}{\eta^2}\left(\begin{array}{l}2\sum_{m=0}^{\beta-2}\frac{C_{\beta-2:m}(-1)^m}{(m+2)^2}+\\ \frac{1}{\eta}\sum_{m=0}^{\beta-3}\frac{C_{\beta-3:m}(-1)^m}{(m+3)^2}-2\sum_{m=0}^{\beta-2}\frac{C_{\beta-2:m}(-1)^m}{(m+1)^2}\end{array}\right) \tag{6.8}$$

Using (6.6), (6.7) and (6.8) we can find the minimum variance bound. By using these equations we can also find the fisher information matrix for finding the variances of these parameters.

7. L-Moments

As discussed in previous section, the alternative measure of distribution of shapes denoted by Hosking (1990), L-moments are expectation of certain linear combinations of order Statistics. Hosking has defined the L-moments of X to be the quantities.

$$\lambda_r=r^{-1}\sum_{k=0}^{i-1}(-1)^k\binom{i-1}{k}E\left(X_{r-k:r}\right) \qquad , \text{r = 1, 2, 3, 4} \tag{7.1}$$

The L in "L-moments" emphasizes that λ_r is a linear function of the expected order statistics. The expectation of an order Statistics has been written as (Hosking (1990))

$$EX_{j:r}=\frac{r!}{(j-1)!(r-j)!}\int x(F)\{F(x)\}^{j-1}\{1-F(x)\}^{r-j}\,dF(x). \tag{7.2}$$

The first few L-moments, λ_r of random variable"X", as defined by Hosking (1990) are given below:

$$\lambda_1=E(X)=\int_0^1 x(F)dF$$

$$\lambda_2=\frac{1}{2}E\left(X_{2:2}-X_{1:2}\right)=\int_0^1 x(F)(2F-1)dF$$

$$\lambda_3=\frac{1}{3}E\left(X_{3:3}-2X_{2:3}+X_{1:3}\right)=\int_0^1 x(F)\left(6F^2 6F+1\right)dF$$

$$\lambda_4=\frac{1}{4}E\left(X_{4:4}-3X_{3:4}+3X_{2:4}-X_{1:4}\right)=\int_0^1 x(F)\left(20F^3-30F^2+12F-1\right)dF$$

Where $X_{k:n}$ is an order Statistics, the k^{th} smallest of a sample of size "n" drawn from the distribution of X and x(F) is a quantile function of real-valued random variable X.

The "L" in L-moment emphasized that λ_r is a linear function of expected order

Statistics. Measures of skewness and kurtosis, based on L-moments, are respectively.

L-skewness = $\tau_3 = \lambda_3/\lambda_2$ (7.3)

And

L-Kurtosis = $\tau_4 = \lambda_4/\lambda_2$ (7.4)

The L-moments $\lambda_1, \cdots, \lambda_r$ and L-moment ratios $\tau_3, \cdots, \tau_r$ are useful quantities for summarizing a distribution. The L-moments are in some ways analogous to the (conventional) center moments and the L-moments ratios are analogous to moment's ratios. In particular $\lambda_1, \lambda_2, \tau_3$ *and* τ_4 may be regarded as measures of location, scale, skewness and kurtosis respectively. Hosking (1990) has shown that L-moments have good properties as measure of distributional shape and are useful for fitting distribution to data.

8. Calculation of L-skewness & L-kurtosis

The probability distribution can be summarized by the following four measures. The mean or L-location (λ_1), The L-scale (λ_2), The L-skewness (τ_3), The L-Kurtosis(τ_4), we now consider these measures, particularly τ_3 and τ_4 in more details.

Moments are often used as summary measure of the shapes of a distribution. In this section the measure of skewness and kurtosis based on L-moments has been calculated from the Generalized Exponential distribution. The first four L-moments of the Discrete Generalized Exponential distribution given in expression (7.1) have been calculated from relations (7.2). The derived first four L-moments of the Discrete Generalized Exponential distribution is explained in eq. (8.1, 8.2, 8.3 & 8.4)

$$\lambda_1 = \beta\eta\sum_{X=1}^{\beta-1} C_{x:\beta-1}\,\frac{(-1)^x}{(1+x)^2} \tag{8.1}$$

$$\lambda_2 = 2\beta\eta\sum_{X=1}^{2\beta-1} C_{x:2\beta-1}\,\frac{(-1)^x}{(1+x)^2} - \beta\eta\sum_{X=1}^{\beta-1} C_{x:\beta-1}\,\frac{(-1)^x}{(1+x)^2} \tag{8.2}$$

$$\lambda_3 = 6\beta\eta\sum_{X=1}^{3\beta-1} C_{x:3\beta-1}\,\frac{(-1)^x}{(1+x)^2} - 6\beta\eta\sum_{X=1}^{2\beta-1} C_{x:2\beta-1}\,\frac{(-1)^x}{(1+x)^2} + \beta\eta\sum_{X=1}^{\beta-1} C_{x:\beta-1}\,\frac{(-1)^x}{(1+x)^2} \tag{8.3}$$

$$\lambda_4 = 20\beta\eta\sum_{X=1}^{4\beta-1} C_{x:4\beta-1}\frac{(-1)^x}{(1+x)^2} - 30\beta\eta\sum_{X=1}^{3\beta-1} C_{x:3\beta-1}\frac{(-1)^x}{(1+x)^2} + 12\beta\eta\sum_{X=1}^{2\beta-1} C_{x:2\beta-1}\frac{(-1)^x}{(1+x)^2} - \beta\eta\sum_{X=1}^{\beta-1} C_{x:\beta-1}\frac{(-1)^x}{(1+x)^2} \tag{8.4}$$

L. Moments of Coefficient of variation, Coefficient of Skewness & Coefficient of Kurtosis

$$C.V_{GED} = \frac{\sqrt{2\beta\eta\sum_{X=1}^{2\beta-1} C_{x:2\beta-1}\frac{(-1)^x}{(1+x)^2} - \beta\eta\sum_{X=1}^{\beta-1} C_{x:\beta-1}\frac{(-1)^x}{(1+x)^2}}}{\beta\eta\sum_{X=1}^{\beta-1} C_{x:\beta-1}\frac{(-1)^x}{(1+x)^2}} \tag{8.5}$$

$$S.K_{GED} = \frac{6\beta\eta\sum_{X=1}^{3\beta-1} C_{x:3\beta-1}\frac{(-1)^x}{(1+x)^2} - 6\beta\eta\sum_{X=1}^{2\beta-1} C_{x:2\beta-1}\frac{(-1)^x}{(1+x)^2} + \beta\eta\sum_{X=1}^{\beta-1} C_{x:\beta-1}\frac{(-1)^x}{(1+x)^2}}{2\beta\eta\sum_{X=1}^{2\beta-1} C_{x:2\beta-1}\frac{(-1)^x}{(1+x)^2} - \beta\eta\sum_{X=1}^{\beta-1} C_{x:\beta-1}\frac{(-1)^x}{(1+x)^2}} \tag{8.6}$$

$$K_{GED} = \frac{20\beta\eta\sum_{X=1}^{4\beta-1} C_{x:4\beta-1}\frac{(-1)^x}{(1+x)^2} - 30\beta\eta\sum_{X=1}^{3\beta-1} C_{x:3\beta-1}\frac{(-1)^x}{(1+x)^2} + 12\beta\eta\sum_{X=1}^{2\beta-1} C_{x:2\beta-1}\frac{(-1)^x}{(1+x)^2} - \beta\eta\sum_{X=1}^{\beta-1} C_{x:\beta-1}\frac{(-1)^x}{(1+x)^2}}{\left(2\beta\eta\sum_{X=1}^{2\beta-1} C_{x:2\beta-1}\frac{(-1)^x}{(1+x)^2} - \beta\eta\sum_{X=1}^{\beta-1} C_{x:\beta-1}\frac{(-1)^x}{(1+x)^2}\right)^2} \tag{8.7}$$

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